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2010 J. Phys. A: Math. Theor. 43 115206

(http://iopscience.iop.org/1751-8121/43/11/115206)

J. Phys. A: Math. Theor. **43** (2010) 115206 (11pp) doi:10.1088/1751-8113/43/11/115206

On a class of reductions of the Manakov–Santini hierarchy connected with the interpolating system

L V Bogdanov

L D Landau ITP, Kosygin str. 2, Moscow 119334, Russia

E-mail: leonid@landau.ac.ru

Received 10 December 2009, in final form 4 February 2010
Published 2 March 2010
Online at stacks.iop.org/JPhysA/43/115206

Abstract
Using the Lax–Sato formulation of the Manakov–Santini hierarchy, we introduce a class of reductions such that the zero-order reduction of this class corresponds to the dKP hierarchy, and the first-order reduction gives the hierarchy associated with the interpolating system introduced by Dunajski. We present the Lax–Sato form of a reduced hierarchy for the interpolating system and also for the reduction of arbitrary order. Similar to the dKP hierarchy, the Lax–Sato equations for L (the Lax function) split from the Lax–Sato equations for M (the Orlov function) due to the reduction, and the reduced hierarchy for an arbitrary order of reduction is defined by Lax–Sato equations for L only. A characterization of the class of reductions in terms of the dressing data is given. We also consider a waterbag reduction of the interpolating system hierarchy, which defines (1+1)-dimensional systems of hydrodynamic type.

PACS numbers: 02.30.Ik, 02.30.Hq, 02.40.Ky

1. Introduction

In this work we construct a class of reductions of the hierarchy associated with the system recently introduced by Manakov and Santini [1] (see also [2, 3]),

$$\begin{aligned} u_{xt} &= u_{yy} + (uu_x)_x + v_x u_{xy} - u_{xx} v_y, \\ v_{xt} &= v_{yy} + u v_{xx} + v_x v_{xy} - v_{xx} v_y, \end{aligned} \tag{1}$$

whose Lax pair is

$$\begin{aligned} \partial_y \boldsymbol{\Psi} &= ((p - v_x)\partial_x - u_x \partial_p)\boldsymbol{\Psi}, \\ \partial_t \boldsymbol{\Psi} &= ((p^2 - v_x p + u - v_y)\partial_x - (u_x p + u_y)\partial_p)\boldsymbol{\Psi}, \end{aligned} \tag{2}$$

where p plays the role of a spectral variable. The Manakov–Santini system is a generalization of the dispersionless KP (Khohlov–Zabolotskaya) equation to the case of general (non-Hamiltonian) vector fields in the Lax pair. For $v = 0$ the system reduces to the dKP equation. Correspondingly, the reduction $u = 0$ gives the equation [4] (see also [5–7])

$$v_{xt} = v_{yy} + v_x v_{xy} - v_{xx} v_y. \tag{3}$$

Using the Lax–Sato formulation of the hierarchy [8–10], we introduce a class of reductions such that the zero-order reduction of this class corresponds to the dKP hierarchy, and the first-order reduction gives a hierarchy associated with the interpolating system introduced in [11], where it was proved that it is 'the most general symmetry reduction of the second heavenly equation by a conformal Killing vector with a null self-dual derivative'. In [11] it was also shown that the interpolating system corresponds to a simple differential reduction $cu = bv_x$ of the Manakov–Santini equation. We present the Lax–Sato form of a reduced hierarchy for the interpolating system and also for the reduction of an arbitrary order. Similar to the dKP hierarchy, the Lax–Sato equations for L (the Lax function) split from the Lax–Sato equations for M (the Orlov function) due to the reduction, and the reduced hierarchy for an arbitrary order of reduction is defined by Lax–Sato equations for L only. In terms of the Manakov–Santini system this class defines differential reductions (not changing the dimension). A characterization of the class of reductions in terms of the dressing data is given. We also consider waterbag-type reductions of the reduced hierarchies (including the interpolating equation hierarchy), which define (1+1)-dimensional systems of hydrodynamic type.

Reductions of the Manakov–Santini system were also considered in the works [12–14], concentrating mostly on (1+1)-dimensional reductions of hydrodynamic type.

2. The Manakov–Santini hierarchy

The Manakov–Santini hierarchy is defined by the Lax–Sato equations [8–10]

$$\frac{\partial}{\partial t_n}\begin{pmatrix} L \\ M \end{pmatrix} = \left(\left(\frac{L^n L_p}{\{L, M\}}\right)_+ \partial_x - \left(\frac{L^n L_x}{\{L, M\}}\right)_+ \partial_p\right)\begin{pmatrix} L \\ M \end{pmatrix}, \tag{4}$$

where L, M, corresponding to the Lax and Orlov functions of the dispersionless KP hierarchy, are the series

$$L = p + \sum_{n=1}^{\infty} u_n(\mathbf{t}) p^{-n}, \tag{5}$$

$$\begin{aligned} M &= M_0 + M_1, \quad M_0 = \sum_{n=0}^{\infty} t_n L^n, \\ M_1 &= \sum_{n=1}^{\infty} v_n(\mathbf{t}) L^{-n} = \sum_{n=1}^{\infty} \tilde{v}_n(\mathbf{t}) p^{-n}, \end{aligned} \tag{6}$$

and $x = t_0$, $\left(\sum_{-\infty}^{\infty} u_n p^n\right)_+ = \sum_{n=0}^{\infty} u_n p^n$, $\left(\sum_{-\infty}^{+\infty} u_n p^n\right)_- = \sum_{-\infty}^{n=-1} u_n p^n$, $\{L, M\} = L_p M_x - L_x M_p$. A more standard choice of times for the dKP hierarchy corresponds to $M_0 = \sum_{n=0}^{\infty} (n+1) t_n L^n$, and it is easy to transfer to it by rescaling of times.

Lax–Sato equations (4) are equivalent to the generating relation [8–10]

$$\left(\frac{\mathrm{d}L \wedge \mathrm{d}M}{\{L, M\}}\right)_- = 0, \tag{7}$$

where the independent variables of the differential include all the times $\mathbf{t}$ and a spectral variable p.

Equations (4) imply that the dynamics of the Poisson bracket $J = \{L, M\}$ is described by the equation [12]

$$\begin{aligned} &\frac{\partial}{\partial t_n} \ln J = (A_n \partial_x - B_n \partial_p) \ln J + \partial_x A_n - \partial_p B_n, \\ &A_n = \left(\frac{L^n L_p}{J}\right)_+, \qquad B_n = \left(\frac{L^n L_x}{J}\right)_+. \end{aligned} \tag{8}$$

This equation together with the first equation of (4) forms a closed system which defines the Manakov–Santini hierarchy and can be used as an equivalent to the system (4), very useful for the description of reductions. Thus, to define the Manakov–Santini hierarchy, it is possible to consider the equations

$$\begin{aligned} &\frac{\partial}{\partial t_n} L = ((L^n L_p J^{-1})_+ \partial_x - (L^n L_x J^{-1})_+ \partial_p) L, \\ &\frac{\partial}{\partial t_n} \ln J = ((L^n L_p J^{-1})_+ \partial_x - (L^n L_x J^{-1})_+ \partial_p) \ln J + \partial_x (L^n L_p J^{-1})_+ - \partial_p (L^n L_x J^{-1})_+ \end{aligned} \tag{9}$$

for the series $L(p)$ (5) and J,

$$J = 1 + \sum_1^\infty j_n(\mathbf{t}) L^{-n} = 1 + \sum_1^\infty \tilde{j}_n(\mathbf{t}) p^{-n}. \tag{10}$$

The function M can be found from L and J using the relation [12]

$$J = \{L, M\} = (\partial_p L) \partial_x M|_L,$$

where $|_L$ means that a partial derivative is taken for fixed L. Then

$$\partial_x M|_L = J(\partial_p L)^{-1} = J \partial_L p(L), \tag{11}$$

and, introducing series for $p(L)$ (inverse to $L(p)$ (5)),

$$p = L + \sum_1^\infty p_n(\mathbf{t}) L^{-n}, \tag{12}$$

it is possible to find coefficients of the series for $\partial_x M|_L$ explicitly and define the function M. For the first coefficient of the series (6) we obtain $\partial_x v_1(\mathbf{t}) = j_1(\mathbf{t})$. In the case of Hamiltonian vector fields $J = 1$ and $\partial_x M|_L = \partial_L p(L)$.

Lax–Sato equations for the first two flows of the hierarchy (4)

$$\partial_y \begin{pmatrix} L \\ M \end{pmatrix} = ((p - v_x)\partial_x - u_x \partial_p) \begin{pmatrix} L \\ M \end{pmatrix}, \tag{13}$$

$$\partial_t \begin{pmatrix} L \\ M \end{pmatrix} = ((p^2 - v_x p + u - v_y)\partial_x - (u_x p + u_y)\partial_p) \begin{pmatrix} L \\ M \end{pmatrix}, \tag{14}$$

where $u = u_1$, $v = v_1$, $x = t_0$, $y = t_1$, $t = t_2$, correspond to the Lax pair (2) of the Manakov–Santini system (1).

Equation (13) gives the recursion relations, defining the coefficients of the series $L(p)$ (5), $M(p)$ (6) through the functions u, v:

$$\partial_x u_{n+1} = \partial_y u_n + v_x \partial_x u_n - (n-1) u_x u_{n-1}, \tag{15}$$

$$\partial_x \tilde{v}_{n+1} - u_n = \partial_y \tilde{v}_n + v_x \partial_x \tilde{v}_n - (n-1) u_x \tilde{v}_{n-1}, \qquad n \geqslant 1, \quad \tilde{v}_1 = v. \tag{16}$$

Using these relations, the Manakov–Santini system can be directly obtained from (14) without the application of compatibility conditions for linear equations. It is also possible to consider equations for $\ln J$ (9): the first two flows read

$$\partial_y \ln J = ((p - v_x)\partial_x - u_x\partial_p) \ln J - v_{xx}, \tag{17}$$

$$\partial_t \ln J = ((p^2 - v_x p + u - v_y)\partial_x - (u_x p + u_y)\partial_p) \ln J - v_{xx} p - v_{xy}, \tag{18}$$

and the recursion relation for $\ln J = \sum_{n=1}^{\infty} (\ln J)_n p^{-n}$ is similar to the recursion for $L(p)$,

$$\partial_x (\ln J)_{n+1} = \partial_y (\ln J)_n + v_x \partial_x (\ln J)_n - (n-1)u_x (\ln J)_{n-1},$$

where $n \geqslant 1$, $(\ln J)_1 = v_x$.

3. A class of reductions connected with the interpolating system

In this section we consider a class of reductions of the Manakov–Santini hierarchy, characterized by the existence of a polynomial solution (of order k in p) of the non-homogeneous linear equation (8). For $k = 0$ this reduction corresponds to Hamiltonian vector fields and the dKP hierarchy. For $k = 1$ we obtain the interpolating system [11] hierarchy. For general k, J can be explicitly expressed through L, and the reduced hierarchy is defined by the Lax–Sato equations for L only (similar to the dKP hierarchy).

Let $\ln J$ satisfy non-homogeneous equations (8) and L satisfy homogeneous equations (4); then the function $\ln J + F(L)$ also satisfies equations (8). We define a class of reductions of the Manakov–Santini hierarchy by the condition

$$(\ln J - \alpha L^k)_- = 0, \tag{19}$$

where α is a constant, which means that equations (8) have an analytic solution $(\ln J - \alpha L^k)$. This condition defines a reduction because A_n, B_n in equations (8) are polynomials, and the dynamics, defined by these equations, preserves analyticity of the functions, so analytic solutions form an invariant manifold. Thus, if $(\ln J - \alpha L^k)(x, p)$ is polynomial with respect to p at the initial point in higher times, then it is polynomial for arbitrary values of higher times.

Reduction (19) is completely characterized by the existence of a polynomial solution of equations (8).

Proposition 1. *The existence of a polynomial solution*

$$f = -\alpha p^k + \sum_{0}^{i=k-2} f_i(\mathbf{t}) p^i$$

(where the coefficients f_i do not contain constants, see below) of equations (8),

$$\frac{\partial}{\partial t_n} f = (A_n \partial_x - B_n \partial_p) f + \partial_x A_n - \partial_p B_n, \tag{20}$$

is equivalent to the reduction condition (19).

Proof. First, the reduction condition (19) directly implies that $f = (\ln J - \alpha L^k)$ is a polynomial solution of equations (20) of the required form; thus, the existence of a polynomial solution is necessary.

To prove that it is sufficient, we note that $F = \ln J - f$ solves homogeneous equations (20) (equations (4)). Let us expand p into the powers of L (12), and represent F in the form

$$F = \alpha L^k + \sum_{-\infty}^{i=k-2} F_i(\mathbf{t}) L^i,$$

where $F_i(\mathbf{t})$ can be expressed through $f_i(\mathbf{t})$ and coefficients of expansion of J and L (respectively, $j_n(\mathbf{t})$ and $u_n(\mathbf{t})$). It is easy to check that F solves homogeneous equations (20) iff all the coefficients $F_i(\mathbf{t})$ are constants. Suggesting that the coefficients f_i of the polynomial $f(p)$ do not contain constants (in the sense that they are equal to zero if all the coefficients $j_n = u_n = 0$), we come to the conclusion that $\ln J - f = \alpha L^k$. □

The simplest case $k = 0$ corresponds to Hamiltonian vector fields. Indeed, in this case $J = 1$, and from equations (8) we have

$$\partial_x A_n - \partial_p B_n = 0.$$

In the case $k = 1$

$$\begin{aligned} &(\ln J - \alpha L)_- = 0 \quad \Rightarrow \quad (\ln J - \alpha L) = (\ln J - \alpha L)_+ = -\alpha p, \\ &J = \exp \alpha (L - p). \end{aligned} \tag{21}$$

So, similar to the case of Hamiltonian vector fields, the equation for L splits off and the reduced hierarchy is defined by the Lax–Sato equations

$$\frac{\partial}{\partial t_n} L = (\mathrm{e}^{\alpha(p-L)} L^n L_p)_+ \partial_x L - (\mathrm{e}^{\alpha(p-L)} L^n L_x)_+ \partial_p L. \tag{22}$$

The generating relation for the reduced hierarchy reads

$$(\mathrm{e}^{\alpha(p-L)} \mathrm{d}L \wedge \mathrm{d}M)_- = 0,$$

or, equivalently,

$$(\mathrm{e}^{-\alpha L} \mathrm{d}L \wedge \mathrm{d}M)_- = 0.$$

Representing relation (21) as a series in p^{-1}, in the first nontrivial order we obtain (see (11))

$$\alpha u = j_1 = v_x, \tag{23}$$

which is exactly the condition used in [11] to reduce the Manakov–Santini system to the interpolating equation ($\alpha = \frac{c}{b}$ in the notation of [11]). The Manakov–Santini system (1) with the reduction (23) is equivalent to the interpolating equation up to a simple transformation, and we will call the hierarchy (22) *the interpolating equation hierarchy*.

The reduction condition (21) implies that $(-\alpha p)$ is a solution of equations (8) (in fact, these conditions are *equivalent*), and, substituting it, we obtain the reduction equations in terms of vector field components:

$$\partial_x A_n - \partial_p B_n - B_n = 0. \tag{24}$$

It is easy to check that for $n = 1$ we obtain the reduction condition (23).

3.1. General k

In the general case

$$\begin{aligned} &(\ln J - \alpha L^k)_- = 0 \quad \Rightarrow \quad (\ln J - \alpha L^k) = (\ln J - \alpha L^k)_+ = -\alpha (L^k)_+, \\ &J = \exp \alpha \big(L^k - (L^k{}_+)\big) = \exp \alpha (L^k{}_-), \end{aligned} \tag{25}$$

and Lax–Sato equations of reduced hierarchy read

$$\frac{\partial}{\partial t_n} L = (\mathrm{e}^{-\alpha(L^k{}_-)} L^n L_p)_+ \partial_x L - (\mathrm{e}^{-\alpha(L^k{}_-)} L^n L_x)_+ \partial_p L. \tag{26}$$

These equations imply equations (9) for J (25), and the function M is defined by relation (11),

$$\partial_x M|_L = J(\partial_p L)^{-1} = \mathrm{e}^{\alpha(L^k - (L^k{}_+))} (\partial_p L)^{-1}.$$

Generating relation (7) in this case takes the form

$$\left(\mathrm{e}^{-\alpha L^k}\mathrm{d}L \wedge \mathrm{d}M\right)_- = 0. \tag{27}$$

Reduction (25) is equivalent to the condition that $(-\alpha L^k{}_+)$ is a solution to equations (8), which gives a differential characterization of the reduction in terms of the Manakov–Santini hierarchy:

$$\begin{aligned}&\frac{\partial}{\partial t_n}\left(\alpha L^k{}_+\right) = (A_n\partial_x - B_n\partial_p)\left(\alpha L^k{}_+\right) - \partial_x A_n + \partial_p B_n,\\ &A_n = \left(\frac{L^n L_p}{J}\right)_+, \qquad B_n = \left(\frac{L^n L_x}{J}\right)_+.\end{aligned} \tag{28}$$

For the first flow $n = 1$ we obtain a condition (compare (17))

$$\partial_y\left(\alpha L^k{}_+\right) = ((p - v_x)\partial_x - u_x\partial_p)\left(\alpha L^k{}_+\right) + v_{xx}. \tag{29}$$

This condition defines a differential reduction of the Manakov–Santini system.

Let us consider in more detail the case $k = 2$. A reduction is defined by relation (25),

$$J = \mathrm{e}^{\alpha(L^2{}_-)}. \tag{30}$$

Taking an expansion into powers of p^{-1}, in the first nontrivial order we obtain

$$j_1 = 2\alpha u_2.$$

Using the recursion formula (15), we obtain

$$\partial_x u_2 = u_y + v_x u_x.$$

Thus, we come to the conclusion that in terms of the Manakov–Santini system (1) reduction (30) leads to a condition

$$2\alpha(u_y + v_x u_x) = v_{xx}. \tag{31}$$

This condition defines a differential reduction of the Manakov–Santini system.

Another way to obtain the reduction is to use relation (29). Indeed, $\left(L^2{}_+\right) = p^2 + 2u$, and, substituting this expression into (29), we obtain

$$2\alpha u_y = 2\alpha((p - v_x)u_x - u_x p) + v_{xx} \Rightarrow 2\alpha(u_y + 2v_x u_x) = v_{xx}.$$

Relation (29) gives differential reductions of arbitrary order k for the Manakov–Santini system in explicit form.

For illustration we will also calculate a differential reduction of the Manakov–Santini system of the order $k = 3$. In this case $\left(L^3{}_+\right) = p^3 + 3pu + 3u_2$, and, substituting this expression into (29), we obtain

$$3\alpha(\partial_y(u_y + u_x v_x) + \partial_x(u_y v_x + u_x v_x^2 + u u_x)) = v_{xxx}. \tag{32}$$

3.2. A pair of reductions with different k—reduction to (1+1)

If we consider a pair of reductions with different k, we obtain a closed (1+1)-dimensional system of equations for the functions u, v. First let us consider reductions of the interpolating system, i.e. the reduction with $k = 1$, which leads to condition (23), together with reduction (19) of some order $k \neq 1$ (with a constant β).

For $k = 2$, using (19) and (31), we obtain a system

$$u_y + v_x u_x = (2\beta)^{-1} v_{xx}, \qquad v_x = \alpha u,$$

which implies a hydrodynamic-type equation (Hopf-type equation) for u:

$$u_y + \alpha u u_x = \frac{\alpha}{2\beta} u_x.$$

The system for $k = 3$ reads (see (32))

$$\partial_y(u_y + u_x v_x) + \partial_x\left(u_y v_x + u_x v_x^2 + u u_x\right) = 3\beta^{-1} v_{xxx}, \qquad v_x = \alpha u,$$

which implies an equation for u,

$$u_{yy} + \partial_x\left(2\alpha u_y u + \alpha^2 u_x u^2 + u u_x - \frac{\alpha}{3\beta} u_x\right) = 0,$$

which can be rewritten as a system of hydrodynamic type for two functions u, w,

$$w_y = \left(\frac{\alpha}{3\beta} - \alpha^2 u^2 - u\right) u_x - 2\alpha u w_x, \qquad u_y = w_x.$$

A system of equations of hydrodynamic type corresponding to the reduction of interpolating system of arbitrary order $k > 3$ can be obtained using the observation that $f = \beta L^k{}_+ - \alpha p$ is a solution of the linear equation

$$\partial_y f = (p - \alpha u)\partial_x f - u_x \partial_p f,$$

which provides a system of hydrodynamic type for the coefficients of the polynomial $f = \beta p^k + k\beta u p^{k-2} - \alpha p + \sum_{i=0}^{k-3} f_i p^i$, namely

$$\begin{aligned}
&\partial_y u = (k\beta)^{-1}\partial_x f_{k-3} - \alpha u \partial_x u,\\
&\partial_y f_{k-3} = \partial_x f_{k-4} - \alpha u \partial_x f_{k-3} - k(k-2)\partial_x u,\\
&\partial_y f_i = \partial_x f_{i-1} - \alpha u \partial_x f_i - (i+1) f_{i+1}\partial_x u, \qquad 0 < i < k-3,\\
&\partial_y f_0 = -\alpha u \partial_x f_0 - (f_1 - \alpha)\partial_x u.
\end{aligned}$$

Let us also consider a simple example of a system defined by two reductions of higher order, taking reductions of order 2 (31) and of order 3 (32),

$$\begin{aligned}
&u_y + v_x u_x = (2\alpha)^{-1} v_{xx},\\
&(\partial_y(u_y + u_x v_x) + \partial_x(u_y v_x + u_x v_x^2 + u u_x)) = (3\beta)^{-1} v_{xxx}.
\end{aligned}$$

This system can be rewritten as a system of hydrodynamic type for the functions u, $w = v_x$:

$$\begin{aligned}
&u_y + w u_x = (2\alpha)^{-1} w_x,\\
&w_y = \frac{2\alpha}{3\beta} w_x - w w_x - 2\alpha u u_x.
\end{aligned}$$

4. A waterbag reduction for the interpolating system hierarchy

For the class of reduced hierarchies defined by Lax–Sato equations (26) it is possible to consider a manifold of solutions of the form

$$L(p, x) = p - \sum_{i=1}^{N} c_i \ln(p - w_i(x)), \qquad \sum_{i=1}^{N} c_i = 0, \tag{33}$$

where c_i are some constants. Due to the fact that coefficients of vector fields in equations (26) are polynomial, and the 'plus' projection of the equations is identically zero by construction, it is straightforward to demonstrate that this manifold is invariant under the dynamics, so it defines a reduction (this type of reduction is known for the dKP hierarchy as a waterbag reduction). Each of Lax–Sato equations (26) in this case is equivalent to the closed (1+1)-dimensional system of equations for the functions w_i.

Let us study in more detail the waterbag reduction for the interpolating equation hierarchy (22). The first two Lax–Sato equations of the hierarchy read

$$\begin{aligned}\partial_y L &= (p-\alpha u)\partial_x L - u_x\partial_p L,\\ \partial_t L &= (p^2-\alpha up-\alpha u_2+u)\partial_x L-(u_x p-\alpha uu_x+\partial_x u_2)\partial_p L.\end{aligned} \tag{34}$$

For the Lax–Sato function (33) the coefficients of expansion u_n are expressed through the functions w_i as

$$u_n=\sum_{i=1}^N \frac{c_i}{n}w_i^n. \tag{35}$$

Substituting the ansatz (33) into the Lax–Sato equations (34) and using (35), we obtain two closed (1+1)-dimensional systems of equations for the functions w_i:

$$\begin{aligned}\partial_y w_i &= \left(w_i-\alpha\sum_{i=1}^N c_i w_i\right)\partial_x w_i+\partial_x\sum_{i=1}^N c_i w_i,\\ \partial_t w_i &= \left(w_i^2-\alpha w_i\sum_{i=1}^N c_i w_i-\alpha\sum_{i=1}^N\frac{c_i}{2}w_i^2+\sum_{i=1}^N c_i w_i\right)\partial_x w_i\\ &\quad+\left(w_i-\alpha\sum_{i=1}^N c_i w_i\right)\partial_x\sum_{i=1}^N c_i w_i+\partial_x\sum_{i=1}^N\frac{c_i}{2}w_i^2.\end{aligned} \tag{36}$$

These systems (as well as higher flows) are compatible, because they are constructed as a reduction of the flows of the Manakov–Santini hierarchy to the invariant manifold (33). On the invariant manifold equations (36) are equivalent to the Lax–Sato equations of the Manakov–Santini hierarchy. Equations (36) are (1+1)-dimensional systems of hydrodynamic type, and their common solution gives a solution of the interpolating system (the Manakov–Santini system (1) with the reduction $\alpha u = v_x$) by the formula

$$u=\sum_{i=1}^N c_i w_i.$$

In the case $\alpha = 0$ equations (36) give the waterbag reduction of the dKP hierarchy [15] (to match (36) to the formulae of the work [15], it is necessary to rescale the times).

The minimal number of components w_i in equations (36) is 2, and for the simplest case $N = 2$, $L(p,x) = p - c\ln\frac{p-w_1(x)}{p-w_2(x)}$, an explicit form of the hydrodynamic-type system corresponding to the first flow of (36) is

$$\begin{aligned}\partial_y w_1 &= \partial_x\left(\frac12 w_1^2+c(w_1-w_2)\right)-\alpha c(w_1-w_2)\partial_x w_1,\\ \partial_y w_2 &= \partial_x\left(\frac12 w_2^2+c(w_1-w_2)\right)-\alpha c(w_1-w_2)\partial_x w_2,\end{aligned}$$

and the second flow reads

$$\begin{aligned}\partial_t w_1 &= \partial_x\left(\frac13 w_1^3+cw_1(w_1-w_2)+\frac{c}{2}(w_1^2-w_2^2)\right)\\ &\quad-\alpha\left(cw_1(w_1-w_2)\partial_x w_1+\frac{c^2}{2}\partial_x(w_1-w_2)^2\right),\\ \partial_t w_2 &= \partial_x\left(\frac13 w_2^3+cw_2(w_1-w_2)+\frac{c}{2}(w_1^2-w_2^2)\right)\\ &\quad-\alpha\left(cw_2(w_1-w_2)\partial_x w_2+\frac{c^2}{2}\partial_x(w_1-w_2)^2\right).\end{aligned}$$

The Zakharov reduction, corresponding to rational L with simple poles, can be considered as a degenerate case of the waterbag reduction, in the limit when pairs of functions w_i coincide. In the two-component case, considering the limit $c \to \infty$, $w_1 - w_2 = c^{-1}u$, we obtain $L = p + \frac{u}{p-w}$, and the equations of the reduced hierarchy can be obtained as a limit of equations for the waterbag reduction. For the first two flows

$$\partial_y w = \partial_x \left(\tfrac{1}{2}w^2 + u\right) - \alpha u \partial_x w,$$
$$\partial_y u = \partial_x (wu) - \alpha u \partial_x u,$$

and

$$\partial_t w = \partial_x \left(\tfrac{1}{3}w^3 + 2wu\right) - \alpha \left(wu\partial_x w + \tfrac{1}{2}\partial_x u^2\right),$$
$$\partial_t u = \partial_x (w^2 u + u^2) - \alpha u \partial_x (wu).$$

A common solution of these systems gives a solution u of the interpolating equation.

5. The characterization of reductions in terms of the dressing data

A dressing scheme for the Manakov–Santini hierarchy can be formulated in terms of two-component nonlinear Riemann–Hilbert problem on the unit circle S in the complex plane of the variable p,

$$L_{\text{in}} = F_1(L_{\text{out}}, M_{\text{out}}), \qquad M_{\text{in}} = F_2(L_{\text{out}}, M_{\text{out}}), \tag{37}$$

where the functions $L_{\text{in}}(p, \mathbf{t})$, $M_{\text{in}}(p, \mathbf{t})$ are analytic inside the unit circle, the functions $L_{\text{out}}(p, \mathbf{t})$, $M_{\text{out}}(p, \mathbf{t})$ are analytic outside the unit circle and have an expansion of the form (5), (6). The functions F_1, F_2 are suggested to define (at least locally) a diffeomorphism of the plane, $\mathbf{F} \in \text{Diff}(2)$, and we call them the dressing data. It is straightforward to demonstrate that the problem (37) implies the analyticity of the differential form

$$\Omega_0 = \frac{\mathrm{d}L \wedge \mathrm{d}M}{\{L, M\}}$$

(where the independent variables of the differential include all the times $\mathbf{t}$ and p) in the complex plane and the generating relation (7), thus defining a solution of the Manakov–Santini hierarchy. Considering a reduction to the group of area-preserving diffeomorphisms SDiff(2), we obtain the dKP hierarchy.

To obtain the interpolating system, it is necessary to consider a more general class of reductions. Let $G_1(\lambda, \mu)$, $G_2(\lambda, \mu)$ define an area-preserving diffeomorphism, $\mathbf{G} \in \text{SDiff}(2)$,

$$\left|\frac{D(G_1, G_2)}{D(\lambda, \mu)}\right| = 1.$$

Let us fix a pair of analytic functions $f_1(\lambda, \mu)$, $f_2(\lambda, \mu)$ (the reduction data) and consider a problem

$$f_1(L_{\text{in}}, M_{\text{in}}) = G_1(f_1(L_{\text{out}}, M_{\text{out}}), f_2(L_{\text{out}}, M_{\text{out}})),$$
$$f_2(L_{\text{in}}, M_{\text{in}}) = G_2(f_1(L_{\text{out}}, M_{\text{out}}), f_2(L_{\text{out}}, M_{\text{out}})), \tag{38}$$

which defines a reduction of the MS hierarchy. In terms of the Riemann problem for the MS hierarchy (37), which can be written in the form

$$(L_{\text{in}}, M_{\text{in}}) = \mathbf{F}(L_{\text{out}}, M_{\text{out}}), \tag{39}$$

the reduction condition for the dressing data reads

$$\mathbf{f} \circ \mathbf{F} \circ \mathbf{f}^{-1} \in \text{SDiff}(2). \tag{40}$$

In terms of equations of the MS hierarchy the reduction is characterized by the condition

$$(\mathrm{d}f_1(L,M)\wedge \mathrm{d}f_2(L,M))_{\text{out}} = (\mathrm{d}f_1(L,M)\wedge \mathrm{d}f_2(L,M))_{\text{in}};$$

thus, the differential form

$$\Omega_{\text{red}} = \mathrm{d}f_1(L,M)\wedge \mathrm{d}f_2(L,M)$$

is analytic in the complex plane, and the reduced hierarchy is defined by the generating relation

$$(\mathrm{d}f_1(L,M)\wedge \mathrm{d}f_2(L,M))_- = 0.$$

Taking

$$f_1(L,M) = L, \qquad f_2(L,M) = \mathrm{e}^{-\alpha L^n} M, \tag{41}$$

we obtain the generating relation

$$\left(\mathrm{e}^{-\alpha L^k}\mathrm{d}L\wedge \mathrm{d}M\right)_- = 0,$$

coinciding with (27). Thus, we come to the following conclusion.

Proposition 2. *In terms of the dressing data for the problem (39), a class of reductions (19) is characterized by condition (40), where* **f** *is defined by (41).*

For the interpolating equation we have $f_1 = L$, $f_2 = \mathrm{e}^{-\alpha L}M$, and the Riemann problem (38) can be written in the form

$$L_{\text{in}} = G_1(L_{\text{out}}, \mathrm{e}^{-\alpha L_{\text{out}}}M_{\text{out}}),$$
$$M_{\text{in}} = \mathrm{e}^{\alpha G_1(L_{\text{out}},\mathrm{e}^{-\alpha L_{\text{out}}}M_{\text{out}})}G_2(L_{\text{out}}, \mathrm{e}^{-\alpha L_{\text{out}}}M_{\text{out}}),$$

where $\mathbf{G} \in \mathrm{SDiff}(2)$.

Acknowledgments

The author is grateful to S V Manakov and M V Pavlov for useful discussions. This research was partially supported by the Russian Foundation for Basic Research under grant nos 06-01-89507 (Russian–Taiwanese grant 95WFE0300007), 08-01-90104, 07-01-00446, 09-01-92439, and by the President of Russia grant 4887.2008.2 (scientific schools).

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